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# THE PERIODIC CONTACT PROBLEM FOR AN ELASTIC HALF-SPACE†

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A method of solving the periodic contact problem for a system of indentors of arbitrary shape and an elastic half-space is proposed. Different versions of the arrangement of the indentors, at one and at several levels, are considered. The results are used to analyse the effect of the parameters of the microgeometry of the characteristics of a discrete contact and the stressed state of solids possessing regular microrelief. © 1999 Elsevier Science Ltd. All rights reserved.

The contact problem in its classical formulation assumes that the surface is ideally smooth and that the contact area is continuous. In fact, the contact area is discrete, due to the existence of surface microrelief. To investigate the effect of the microrelief on the stress–strain state of the surface layers of solids in contact interaction, we need to solve the problem of multiple contact, i.e. the mixed problem of the mechanics of a deformed solid for a system of contact spots, comprising the actual contact area of the surfaces having the microrelief. Numerical methods are usually employed to solve this problem [1, 2], in which case the error in determining the stress–strain state of solids depends on the accuracy with which the function describing the geometry of the surfaces of the contacting solids is specified, and the accuracy of the computational methods employed.

In the mechanics of contact interaction between rough solids, modelling of the rough surface of a system of spherical segments of the same radius (roughness), the height of which is taken to be a random quantity having a certain distribution law, is widely used to calculate the characteristics of the discrete contact. It is assumed that each asperity is elastically deformed in accordance with Hertz's theory. The effect of the other asperities is estimated by means of an average (nominal) pressure [3, 4]. As will be shown below, this approach may lead to calculation errors at high contact densities, when the penetration of an individual asperity depends considerably on the distribution of the pressures on the contact spots in its neighbourhood.

For surfaces with regular relief (for example, a wavy surface) methods of solving periodic contact problems can be employed to investigate the stressed state. Periodic contact problems for elastic solids were considered in the plane formulation in [5–8], and also for a surface having a sinusoidal waviness in two mutually perpendicular directions in [9].

In this paper we present a solution of the periodic contact problem for a system of elastic indentors of arbitrary shape, which model the microgeometry of one of the contacting surfaces, and an elastic half-space, by means of which we analyse the effect of the parameters of the surface geometry on the characteristics of the discrete contact (the pressures and the contact area) and the stress-concentration points in the surface layers of interacting solids.

## 1. FORMULATION OF THE PROBLEM FOR A SINGLE-LEVEL SYSTEM OF INDENTORS

Consider a system of similar axisymmetrical elastic indentors, the shape of the contacting surfaces of which is described by the function $z = f(r)$, which interact without friction with an elastic half-plane (Fig. 1). The axes of the indentors are perpendicular to the boundary of the half-space $z = 0$, while their points of intersection with the boundary are uniformly distributed in the $z = 0$ plane and have coordinates $(r_i, \theta_{ij})$ $(i = 1, 2, \ldots; j = 1, 2, \ldots m_i$, where $m_i$ is the number of indentors located on a circumference of radius $r_i$, $r_i < r_{i+1})$.

To determine the pressure distribution on an arbitrary contact spot we will use the solution of the contact problem, obtained by Galin [10], of the penetration of an axisymmetric punch $(z = f(r))$ into an elastic half-space when a specified load $q(r, \theta)$ acts on the boundary of the half-spaces outside the

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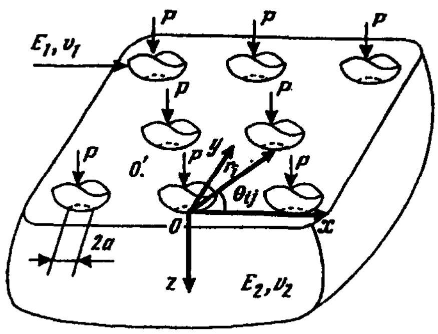


Fig. 1.

punch. The expression for the pressure $p(r, \theta)$ inside the contact area $r \leqslant a$, extend to the case of the contact between two elastic solids, has the form

$$p(r, \theta)=G(r)+\frac{c(\theta)}{\sqrt{a^2-r^2}}-\int_a^{+\infty}\int_0^{2\pi} q(r', \theta')H_2(r, \theta, r', \theta')r'dr'd\theta' \tag{1.1}$$

where

$$G(r)=\frac{E^*}{4\pi^2}\int_0^a \Delta f(r')H_1(r, r')dr' \tag{1.2}$$

$$H_1(r, r')=\int_0^{2\pi} \frac{2r'}{R(r, r', \theta')}\operatorname{arctg}\frac{\sqrt{a^2-r^2}\,\sqrt{a^2-r'^2}}{aR(r, r', \theta')}d\theta'$$

$$H_2(r, \theta, r', \theta')=\frac{\sqrt{r'^2-a^2}}{\pi^2\sqrt{a^2-r^2}\,R^2(r, r', \theta-\theta')}$$

$$R(r, r', \theta')=\sqrt{r^2-2rr'\cos\theta'+r'^2}, \quad E^*=\left(\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}\right)^{-1}$$

and $E_1$, $\nu_1$ and $E_2$, $\nu_2$ are the moduli of elasticity of the materials of the indentors and of the half-space respectively. The function $c(\theta)$ depends on the indentor shape $f(r)$. For a smooth indentor (the function $f'(r)$ is continuous when $r \leqslant a$), in view of the condition that the pressure at the edge of the contact area should be zero, i.e. $p(a, \theta) = 0$, the function $c(\theta)$ has the form

$$c(\theta)=\int_a^{+\infty}\int_0^{2\pi} q(r', \theta')H_2(a, \theta, r', \theta')r'dr'd\theta' \tag{1.3}$$

Taking into account the fact that, in the periodic problem considered, the load is produced by the same indentors, and assuming that the pressure under each indentor is distributed inside a circular contact area of radius $a$, we obtain the following integral equation for determining the contact pressure $p(r, \theta)$

$$p(r, \theta)-\int_0^a\int_0^{2\pi} K(r, \theta, r', \theta')p(r', \theta')r'dr'd\theta'=G(r) \tag{1.4}$$

where

$$K(r, \theta, r', \theta')=\sum_{i=1}^{\infty} K_i(r, \theta, r', \theta') \tag{1.5}$$

$$K_i(r, \theta, r', \theta')=\frac{1}{\pi^2\sqrt{a^2-r^2}}\sum_{j=1}^{m_i} [K_{ij}(a, \theta, r', \theta')-K_{ij}(r, \theta, r', \theta')] \tag{1.6}$$

$$K_{ij}(r,\ \theta,\ r',\ \theta') = L(r_i,\ r',\ \theta_{ij} - \theta')\lambda(r,\ \theta,\ r',\ \theta',\ r_i,\ \theta_{ij})$$

$$L(r_i,\ r',\ \varphi) = \sqrt{r_i^2 + r'^2 + 2r_i r' \cos\varphi - a^2}$$

$$\lambda(r,\ \theta,\ r',\ \theta',\ x,\ \varphi) = [(r\cos\theta - r'\cos\theta' - x\cos\varphi)^2 + (r\sin\theta - r'\sin\theta' - x\sin\varphi)^2]^{-1}$$

In deriving relation (1.4) we assumed that the indentors have a smooth shape and, consequently, $p(a, \theta) = 0$ (the radius $a$ of the contact area in this case is not known in advance).

Note, however, that similar considerations also apply to punches with a fixed size of the contact area (for example, cylinders with a flat base). As a result, an equation will be obtained which is identical in its structure with Eq. (1.4).

The kernel $K(r, \theta, r', \theta')$ of integer equation (1.4) can be represented in the form of the infinite series (1.5). The general term (1.6) of this series can be converted to the form

$$K_i(r,\ \theta,\ r',\ \theta') = \frac{1}{\pi^2\sqrt{a^2 - r^2}} \sum_{j=1}^{m_i} \{2(a-r)\cos(\theta_{ij} - \theta) r_i^{-2} +$$

$$+ (a-r)[-a - r - 6r'\cos(\theta_{ij} - \theta')\cos(\theta_{ij} - \theta) + 2r'\cos(\theta' - \theta)] r_i^{-3} + O(r_i^{-4})\}$$

Since, in the case of the periodic problem, for any indentor situated at the point $(r_i,\ \theta_{ij})$, there is one situated symmetrically to it at the point $(r_i,\ \pi + \theta_{ij})$, the first term in the braces is equal to zero. Since $m_i \sim r_i$, the general term of series (1.5) is of the order of $r^{-2}$ and, consequently, this series converges.

## 2. THE LOCALIZATION METHOD

Together with integral equation (1.4) we will consider the equation

$$p(r,\ \theta) - \int_0^a \int_0^{2\pi} \sum_{i=1}^{n} K_i(r,\ \theta,\ r',\ \theta') p(r',\ \theta') r' dr' d\theta' = G(r) + \bar{N} P Q(r,\ A_n) \qquad (2.1)$$

$$P = \int_0^a \int_0^{2\pi} p(r,\ \theta) r dr d\theta;\quad Q(r,\ A_n) = \frac{2}{\pi} \operatorname{arctg} \frac{\sqrt{a^2 - r^2}}{\sqrt{A_n^2 - a^2}}$$

where $\bar{N}$ is the average number of contact spots per unit area, $P$ is the load acting on each contact spot, and the meaning of the quantity $A_n$ will be explained below.

Equation (2.1) is obtained from (1.4) by replacing the summation when $i > n$ in (1.5) by integration over the area $(\Omega_n : r_i \geqslant A_n,\ 0 \leqslant \theta_{ij} \leqslant 2\pi)$, taking into account the fact that the centres of the contact spots $(r_i,\ \theta_{ij})$ are uniformly distributed over $\Omega_n$. In fact

$$J_n = \sum_{i=n+1}^{\infty} K_i(r,\ \theta,\ r',\ \theta') \approx \bar{N} \int_{A_n}^{+\infty} \int_0^{2\pi} \frac{L(x,\ r',\ \varphi - \theta')}{\pi^2\sqrt{a^2 - r^2}} \times$$

$$\times [\lambda(a,\ \theta,\ r',\ \theta',\ x,\ \varphi) - \lambda(r,\ \theta,\ r',\ \theta',\ x,\ \varphi)] x dx d\varphi$$

Making the replacement of variables

$$y\cos\psi = x\cos\varphi + r'\cos\theta'$$

$$y\sin\psi = x\sin\varphi + r'\sin\theta'$$

and bearing in mind the fact that $r' \leqslant a \ll A_n$, we finally obtain

$$J_n \approx \frac{\bar{N}}{\pi^2\sqrt{a^2 - r^2}} \int_{A_n}^{+\infty} \int_0^{2\pi} \sqrt{y^2 - a^2} \left[\frac{1}{R^2(a,\ y,\ \psi)} - \frac{1}{R^2(r,\ y,\ \psi)}\right] y dy d\psi =$$

$$= \bar{N} Q(r,\ A_n),\quad A_n^2 = \frac{1}{\pi\bar{N}}\left(\sum_{i=1}^{n} m_i + 1\right)$$

($A_n$ is the radius of the circle in which $m_1 + m_2 + \ldots + m_n + 1$ central indentors are situated).

Note that, owing to the choice of $n$, the solution of Eq. (2.1) can approximate as closely as desired to the solution of the initial equation (1.4).

We will consider the structure of Eq. (2.1) in more detail. The integral term on the left-hand side of Eq. (2.1) takes into account the effect on the pressure distribution at a fixed contact spot of the pressures on the contact spots lying close to it (the short-range effect). The effect of the load, distributed over the more distant contact spots, is taken into account by the second term on the right-hand side, which describes the additional pressure in the circular region ($r \leq a$) when a nominal pressure $\bar{p} = P\bar{N}$ acts outside it (in the region $r > A_n$). It follows in fact from relations (1.1) and (1.3) that if the pressure is distributed uniformly outside a circle of radius $A_n$, i.e. $q(r, \theta) = \bar{p}$, it produces on the contact area ($r \leq a$) of the indentor with the elastic half-space an additional pressure

$$p_a(r) = \bar{p}Q(r, A_n)$$

Hence, in periodic contact problems the contact pressure on contact spots far from the one considered (in the region $\Omega_n$) can be taken into account with a certain degree of accuracy by considering the nominal pressure $\bar{p}$ in this region.

This result is a special case of a more general assertion, which we will call the localization method: under conditions of multiple contact the stress–strain state of interacting solids in the region of an individual contact spot can be determined, with a fair degree of accuracy, by taking into account the contact conditions on the contact spot considered and on those contact spots lying close to it (in the local neighbourhood of the spot), and the surface-averaged (nominal) pressure on the remaining part of the interaction region (the nominal contact area). The correctness of this assertion has also been confirmed in [11] in an investigation of the problem of multiple contact with a limited nominal contact region.

Relations (2.1) are used to determine the pressure $p(r, \theta)$ on each contact spot and the radius $a$ of the contact spot. Then, using the known pressures at the boundary of the elastic half-space, the stressed state in the subsurface region is determined. To determine the stresses in the half-space one can use the Boussinesq solution as Green's function (see, for example, [12]).

## 3. A DIVERSE-LEVEL SYSTEM OF INDENTORS

The method of solving periodic contact problems for an elastic half-space proposed above can be used to investigate the contact characteristics when indentors of various heights penetrate into an elastic half-space. Suppose the shapes of the contacting surfaces of the indentors are described by smooth functions $z = f_m(r) + h_m$, where the quantity $h_m (m = 1, 2, \ldots k)$ specifies the height of each level of the system of indentors, and $k$ is the number of levels. We will assume that the contact spot at the $m$-th level is a circle of radius $a_m$. An example of the arrangement at nodal points of an hexagonal array of indentors of each level for $k = 3$ is shown in Fig. 2(a).

We fix an arbitrary contact spot of an $m$-th level indentor and we place the origin of a polar system of coordinates at its centre (see Fig. 2b). Using the localization method, we focus our attention on the distribution of the pressure $p_j(r, \theta)$ ($j = 1, 2, \ldots, k$) on all the contact spots inside a circle ($r \leq A_m$), where

$$A_m^2 = \frac{1}{\pi}\left(\sum_{j=1}^{k} \frac{k_{jm}}{\bar{N}_j} + \frac{1}{\bar{N}_m}\right)$$

where $\bar{N}_j$ is the density of the arrangement of $j$th level indentors, defined as the number of indentors per unit area, and $k_{jm}$ is the number of $j$th level indentors inside the circle ($r \leq A_m$) (when $j = m$ the number of indentors is $k_{mm} + 1$). Replacing the actual pressures at distant contact spots ($r_i > A_m$) by the nominal pressure $\bar{p}$ acting in the region ($r_i > A_m$), where

$$\bar{p} = \sum_{j=1}^{k} \bar{N}_j \int_0^{a_j} \int_0^{2\pi} p_j(r, \theta) r dr d\theta$$

by analogy with (2.1) we obtain the relation

$$p_m(r, \theta) - \sum_{j=1}^{k} \int_0^{a_j} \int_0^{2\pi} K_{jm}(a_m, r, \theta, r', \theta') p_j(r', \theta') r' dr' d\theta' =$$

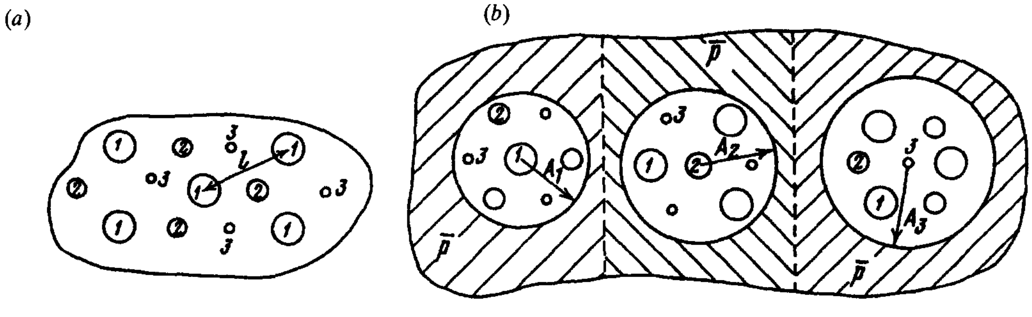


Fig. 2.

$$= G_m(r) + \frac{2\bar{p}}{\pi}\operatorname{arctg}\frac{\sqrt{a_m^2 - r^2}}{\sqrt{A_m^2 - a_m^2}} \tag{3.1}$$

The kernel of Eq. (3.1) has the form

$$K_{jm}(a_m, r, \theta, r', \theta') = \sum_{i=1}^{n_{jm}} K_i(r, \theta, r', \theta')$$

The functions $K_i(r, \theta, r', \theta')$ and $K_{ij}(r, \theta, r', \theta')$ are given by (1.6), in which we must put $a = a_m$; $n_{jm}$ is the number of layers of the $j$th level indentors inside the circle of radius $A_m$. The function $G_m$ is given by relation (1.2) in which $a = a_m$ and $f = f_m$.

Writing relations (3.1) for the indentors of each $m$th level, we obtain a system of integral equations for determining the unknown contact pressures $p_m(r, \theta)$ $(m = 1, 2, \ldots, k)$.

The unknown radii $a_m$ of the contact spots are determined from the specified heights of the indentors $h_m$ from the formula

$$h_m = \frac{1}{\pi E^*}\left[\int_0^{a_m}\int_0^{2\pi} p_m(r, \theta)drd\theta + 2\pi\bar{p}(A_\infty - A_m) + \sum_{j=1}^{k}\sum_{i=1}^{k_{jm}}\int_0^{a_j}\int_0^{2\pi}\frac{p_j(r, \theta)rdrd\theta}{R(r, r_{ij}, \theta - \theta_{ij})}\right] \tag{3.2}$$

where $r_{ij}$, $\theta_{ij}$ are the coordinates of the centres of the indentors of all levels, situated inside the region $(a_m < r_{ij} < A_m, 0 < \theta_{ij} < 2\pi)$. To eliminate the constant $A_\infty$, the system of equations (3.2) is set up for differences in the heights of the indentors $h_1 - h_m$, where $h_1$ is the height of the highest indentor. To close the system of equations (3.2) we use the equilibrium equation

$$\bar{p}\pi A_m^2 = \sum_{j=1}^{k} k_{jm}\int_0^{a_j}\int_0^{2\pi} p_j(r, \theta)rdrd\theta + \int_0^{a_m}\int_0^{2\pi} p_m(r, \theta)rdrd\theta \tag{3.3}$$

It should be noted that for specified $h_m$ all the indentors of the system participate in the contact only for a certain value of the nominal pressure $\bar{p}^*$. When $\bar{p} < \bar{p}^*$ a smaller number of levels of the system of indentors will participate in the contact.

## 4. ANALYSIS OF THE STRESSED STATE

We will use the relations obtained above to analyse the contact characteristics and the stressed state of an elastic half-space under multiple contact conditions. Particular attention will be given to investigating the effect of the geometrical parameter, related to the indentor density, on the characteristics being investigated, which enables us to estimate the limits of applicability of the simplified theories, ignoring the interaction between individual contact spots.

Numerical calculations were carried out for a system of spherical indentors ($f(r) = r^2/(2R)$ where $R$ is the radius of curvature of an indentor), placed at the nodal points of an hexagonal mesh with pitch $l$. For a diverse-level system of indentors we took $k = 3$ (Fig. 2).

We introduce the following dimensionless functions and parameters

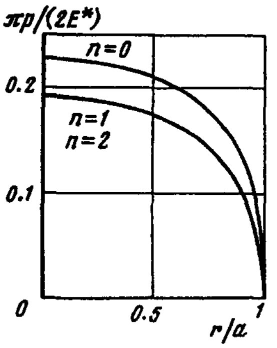


Fig. 3.

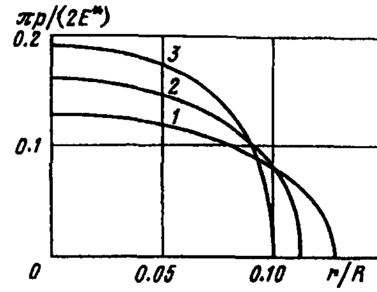


Fig. 4.

$$\rho=\frac{r}{R},\quad A_n'=\frac{A_n}{R},\quad a'=\frac{a}{R},\quad l'=\frac{l}{R},\quad p'(\rho,\ \theta)=\frac{\pi p(\rho R,\ \theta)}{2E^*},\quad P'=\frac{\pi P}{2E^*R^2}$$

The systems of equations (2.1) for the single-level model and (3.1)–(3.3) for a diverse-level model were solved by iteration. The density of arrangements of the indentors of each level was calculated from the formula

$$\overline{N}_j=2/(l^2\sqrt{3})$$

To estimate the accuracy of the localization method and to choose the value of $n$ which gives an error of the results to within a specified degree of accuracy, we calculated the contact pressure $p'(\rho, \theta)$ for a single-level system of indentors with a different value of $n$, characterizing the number of layers of indentors lying close to the indentor considered, on the contact spots of which the pressure distribution is taken into account. Thus, for $n = 1$ we take into account the pressures, distributed over the contact spots, a distance $l$ from the spot considered (six spots, one layer), for $n = 2$ we take into account 12 spots, a distance $l$ from the spot considered (the first layer) and $l\sqrt{3}$ (the second layer), etc. The results of the calculations for $a' = 0.1$ and $1' = 0.2$ ($a/l = 0.5$, which corresponds to the limiting case of close contact) and $n = 0, n = 1$ and $n = 2$, are shown in Fig. 3. The results show that the pressure distributions, calculated for $n = 1$ and $n = 2$, differ by less than 0.1%. This estimate improves as $a/l$ increases. Hence, as a rule we took $n = 1$ for further calculations.

Figure 4 shows the contact pressure under an individual indentor, which is under a load $P' = 0.0044$ when $l' = 1$ (curve 1), $1' = 0.25$ (curve 2) and $1' = 0.2$ (curve 3) for a single-level system of indentors. The results show that, as the distance $l$ between the indentors is reduced, the radius of an individual contact spot is reduced and the maximum pressures on the contact spots increase, and the contact density, characterized by the parameter $a/l$, increases ($a/l = 0.128$ (curve 1), $a/l = 0.45$ (curve 2) and $a/l = 0.5$ (curve 3)). Curve 1 practically coincides with the pressure distribution calculated using Hertz's theory,

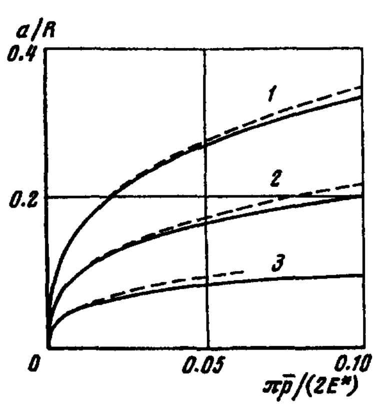


Fig. 5.

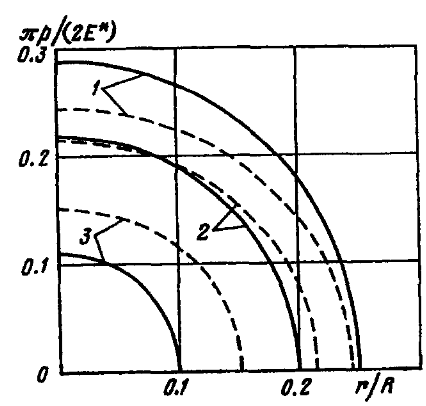


Fig. 6.

so that we can conclude that for small values of $a/l$ one can neglect the mutual influence of the contact spots when calculating the real pressures.

Curves of the radius of a contact spot against the dimensionless nominal pressure $\bar{p}' = \bar{p}\pi/(2E^*)$, drawn for $1' = 1$, (curve 1), $1' = 0.5$ (curve 2) and $1' = 0.2$ (curve 3), are shown in Fig. 5. For comparison the dashed curves show the corresponding results obtained using Hertz's theory. The calculations show that for a constant nominal pressure $\bar{p}$, as the relative distance between the indentors $l/R$ decreases, the radius of an individual contact spot and, consequently, the contact area, also decreases. We can conclude from a comparison with the Hertz curves that for $a/l < 0.25$, the difference in the contact area obtained using the method proposed here and using Hertz's theory does not exceed 2.5%. For larger nominal pressures and, consequently, high contact densities this difference becomes extremely large. Thus, for $1' = 0.5$ (curves 2) and $a/l = 0.44$, the error in calculations using Hertz's theory is 15%.

It is of interest to investigate the contact characteristics for a diverse-level system of indentors, since, in view of the mutual effect of the indentors, taken into account in this model, the instant each new level of indentors enters into contact is determined not only by the nominal pressure and height of an indentor, but also by the density of the contact spots, which have a considerable effect on the curvature of the half-space boundary between contact spots. We made calculations for a system of indentors with a fixed difference between the heights of the different levels: $(h_1 - h_2)/R = 0.014$ and $(h_1 - h_3)/R = 0.037$. In Fig. 6 we show graphs of the distribution of the pressure on the contact spots of each level for an overall load on three indentors of 0.059. The continuous curves 1, 2, and 3 are drawn on the basis of the solution of the periodic problem for indentors of each level with heights of $h_1$, $h_2$, $h_3$ respectively, while the dashed curves were obtained using Hertz's theory. The calculations show that the lower the height of the indentor the more the radius of the contact spot and the pressure distribution on it differ from the corresponding results of Hertz's theory.

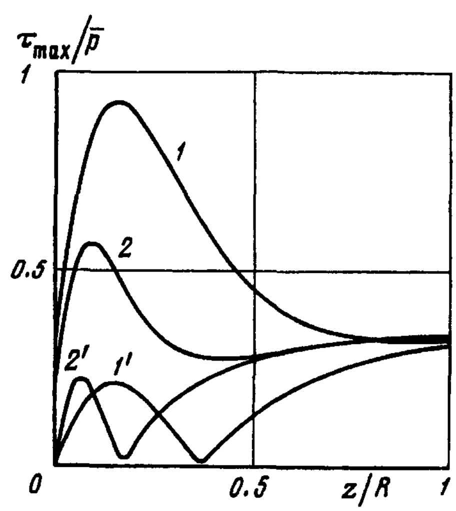


Fig. 7.

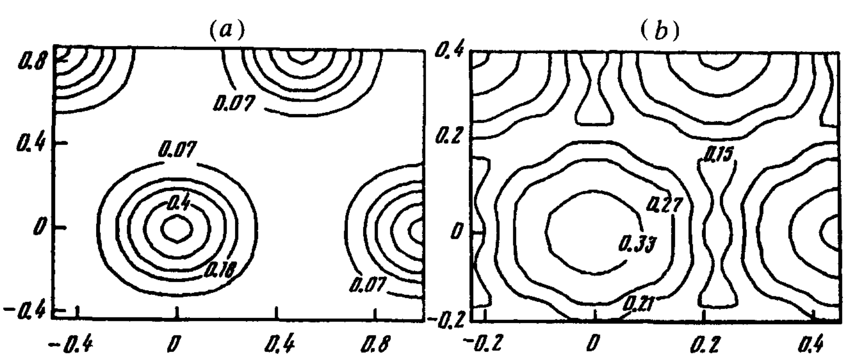


Fig. 8.

An investigation of the stressed state inside the elastic half-space when it interacts with the system of single-level indentors showed that an increase in the stresses occur in a surface layer whose thickness is comparable with half a period, and the value of the stresses in this layer depend very much on $a/l$. In Fig. 7 we show graphs of the principal sheer stresses $\tau_{max}/\bar{p}$ as a function of the depth $z/R$, calculated for different values of $l'$ and $\bar{p}' = 0.12$. Curves 1, $l'$ and 2, 2' are drawn for $l' = 1$ ($a' = 0.35$) and $l' = 0.5$ ($a' = 0.21$) respectively, on the $Oz$ axis, passing through the centre of a contact spot (curves 1 and 2), and along the $O'z$ axis (see Fig. 1), passing through the centre of the unloaded zone (curves 1' and 2'). The calculations showed that the internal stresses only depend appreciably on the contact density parameter $a/l$ for fairly large relative dimensions of the contact spot $0.25 < a/l \leqslant 0.5$. When $a/l$ increases, the maximum values of the principal shear stresses fall, while the point at which they are reached approaches the boundary. In this case the drop in the stresses at a fixed depth decreases. The limiting values of the stresses as $z \to \infty$ are determined by the value of the nominal contact pressures $\bar{p}$.

In Fig. 8 we show isolines of the function $\tau_{max}/\bar{p}$ in the $Oxy$ plane, at a depth of $z/R = 0.08$, where the principal shear stresses are close to their greatest values. The isolines are drawn in a section of the plane ($-l'/2 < x < l'$, $-l'\sqrt{(3)}/4 < y < l'\sqrt{(3)}/2$) for the case when $a' = 0.2$ and $1' = 1$ (Fig. 8a), and $1' = 0.4$ (Fig. 8b). The results show that when the contact density increases the value of principal shear stresses at a fixed depth change only slightly. A similar conclusion can be reached with respect to all the stress components.

Hence, an increase in the contact density leads to the occurrence of a stressed subsurface layer at a certain depth. The stress concentration in this layer may lead to the development of plastic deformations and the microcrack formation in it. The results obtained agree qualitatively with the conclusions reached in [8] when investigating the contact interaction of a sinusoidal punch with an elastic half-plane.

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